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Entropy of the Quantum Scalar Field in Static Black Holes

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The quantum corrections to the entropy of static black holes are investigated by two methods: the brick wall method of 't Hooft and the Euclidean path integral approach of Gibbons and Hawking. Two general formulas for the entropy are obtained and some examples are considered. It is shown that if the contribution from the vacuum surrounding the system is ignored, then the two approaches give the same results. It is found that the entropy of the quantum scalar field in a general static black hole consists of two parts: a quadratically divergent term which takes a geometric character and a logarithmically divergent term which is not proportional to the horizon area. The logarithmically divergent term, in general, depends on the black hole characteristics (in particular, the whole entropy is determined only by this term for some extreme cases) and therefore cannot be neglected as a nonessential additive constant. The renormalization of the entropy is also discussed briefly.

1. INTRODUCTION

One of the interesting problems of black hole physics is to investigate whether the geometric character of the Bekenstein–Hawking entropy (Bekenstein, 1972, 1973; Hawking, 1975) of a black hole [i.e., the entropy of a black hole is proportional to the area of the horizon in four-dimensional spacetime (Bekenstein, 1974; Kallosh *et al.*, 1993; Jing, 1993, 1994)] remains valid when quantum corrections [say, due to quantum fluctuations of the matter fields in the black hole background ('t Hooft, 1985; Susskind and Uglum, 1994)] are taken into account. There has been considerable interest in the problem ('t Hooft, 1985; Susskind and Uglum, 1994; Solodukhin, 1995a–c; Ghosh and Mitra, 1994; Frolov *et al.*, 1996; Kabat and Strassler, 1994; Jing, 1997; Lee and Kim, 1996). 't Hooft

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(1985) studied the quantum corrections to the entropy of the Schwarzschild black hole arising from a scalar field propagating in the region just outside the horizon, by using the brick wall method (BWM). He found that the quantum scalar field theory fluctuations at the Hartle–Hawking temperature give the following leading one-loop contributions to the entropy: $S_q = S_{\text{volume}} + S_{\text{horizon}}$, with $S_{\text{volume}} = 8\pi^3 V/135\beta^3$ and $S_{\text{horizon}} = (8\pi^3/45)(2M)^4/h\beta^3$, where V is the box volume, β is the Hawking inverse temperature, and h is a cutoff. The S_{horizon} takes the geometric character (Solodukhin, 1995b) $A_h/48\pi\varepsilon^2$ if we let $\delta^2 = 2\varepsilon^2/15$ [where $\delta = 2\sqrt{r_h h}$ is the proper distance from the horizon r_h to $r_h + h$, and ε is the ultraviolet cutoff (Solodukhin, 1995b). Ghosh and Mitra (1994) studied the entropy of a scalar field in the background of a dilatonic black hole using BWM. Their result also possesses the geometric character $S_{\text{horizon}} = A_h/48\pi\varepsilon^2$ if we take $\delta^2 = 2\varepsilon^2/15$ for the nonextremal black hole.

Solodukhin (1995a), starting from the one-loop effective action for scalar matter, investigated the quantum corrections to the Schwarzschild black hole by using the Euclidean path integral approach (EPIA) of Gibbons and Hawking (1977; Hawking, 1979). In addition to the 't Hooft term S_{horizon} given above, the Solodukhin entropy also has a logarithmically divergent term $S_{\log} = (1/45)\ \ln(\Lambda/\varepsilon)$, where Λ is the infrared cutoff (Solodukhin, 1995a). When we study the Schwarzschild black hole carefully we note that a logarithmically divergent term exists, in the BWM which can be also cast into the form $S_{\log} = (1/45)\ \ln(\Lambda/\varepsilon)$. We might assume that this term is not essential and does not influence the physics since the entropy is defined up to an arbitrary additive constant. However, for the Reissner–Nordström black hole, Solodukhin (1995a) showed that the appearance of the nongeometric, logarithmically divergent term is typical in four-dimensional spacetime.

In this paper, we study quantum corrections to the entropy of some well-known static black holes by using the BWM and EPIA, and to investigate whether the two methods give the same results and whether the geometric character of the entropy remains valid when quantum corrections are taken into account.

The paper is organized as follows: In Section 2 we first use the BWM to deduce a quantum correction formula for the Bekenstein–Hawking entropy for general static black holes. We then use the expression to study the quantum entropy of some static black holes. In Section 3, by using the EPIM, we generalize Solodukhin's (1995b) equation (15) to general static black holes and use the formula to calculate the quantum entropy of the black holes given in Section 2. The results obtained by the two approaches are then compared. We conclude with some discussions in Section 4.

2. THE ENTROPY OF QUANTUM SCALAR FIELD IN STATIC BLACK HOLES OBTAINED BY THE BRICK WALL METHOD

We now try to find a general entropy expression for a minimally coupled quantum scalar field in thermal equilibrium at temperature $1/\beta$ in static black holes by using the BWM ('t Hooft, 1985). The partition function is given by

$$Z = \sum_{n_q} \exp[-\beta(E_q)n_q] \tag{1}$$

where q denotes a quantum state of the field with energy E_q. The free energy is given by ('t Hooft, 1985)

$$F = \frac{1}{\beta} \int_0^{\infty} dg(E) \ln \{1 - \exp[-\beta E]\} \tag{2}$$

where $g(E) \equiv \pi N$, and N is the total number of waves with energy not exceeding E.

In the BWM ('t Hooft, 1985) the wave function is cut off just outside the horizon, i.e., $\phi = 0$ at $r = r_+ + h$, where h is a small, positive quantity and signifies an ultraviolet cutoff. To remove the infrared divergence we also introduce an infrared cutoff $L >> r_+$ such that $\phi = 0$ for $r = L$.

The wave equation for the scalar field reads

$$\frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - \mu^2 \phi = 0 \tag{3}$$

Using the WKB approximation with $\phi = \exp[-iEt + is(r, \theta, \varphi)]$, we can let $p_r = \partial_r s$, $p_\varphi = \partial_\varphi s$, and $p_\theta = \partial_\theta s$. Substitution of the general static black hole metric

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 \tag{4}$$

into equation (3) yields (Mann *et al.* 1992) $p_r^2 = (1/g^{rr})(-g^{tt}E^2 - g^{\varphi\varphi}p_\varphi^2 - g^{\theta\theta}p_\theta^2 - \mu^2)$. Therefore, in phase space we obtain the number of modes (Padmanabhan, 1986, 1989)

$$\Gamma(E) = \frac{1}{\pi} \int d\varphi \, d\theta \int_{r_+ + h}^{L} dr \times \int dp_\theta \, dp_\varphi \left(\frac{1}{g^{rr}} (-g^{tt}E^2 - g^{\varphi\varphi}p_\varphi^2 - g^{\theta\theta}p_\theta^2 - \mu^2) \right)^{1/2} \tag{5}$$

The integral is taken only over those values for which the square root exists.

We find that the function $g(E) \equiv \pi N$ in 't Hooft (1985) is equal to $(1/8\pi^2)\Gamma(E)$ for static spacetime. Putting the result into equation (2) and then integrating over p_φ and p_θ yields

$$F = -\frac{1}{6\pi^2\beta}\int d\varphi\, d\theta \int_{r_+ + h}^{L} dr \int_{\mu\sqrt{-g_{tt}}}^{\infty} \frac{dE}{e^{\beta E}-1}\frac{\sqrt{-g}}{\sqrt{-g_{tt}}}\left(\frac{E^2}{-g_{tt}} - \mu^2\right)^{3/2} \tag{6}$$

where g is the determinant of the metric $g_{\mu\nu}$. For a massless scalar field, i.e., $\mu = 0$, after carrying out integration of E, the free energy then reduces to

$$F = -\frac{\pi^2}{90\beta^4}\int d\varphi\, d\theta \int_{r_+ + h}^{L} \frac{\sqrt{-g}}{g_{tt}^2}\, dr \tag{7}$$

which is also an approximative result for the case $\mu^2 << r_+/h\beta^2$ and $L >> r_+$ ('t Hooft, 1985). Inserting equation (7) into the formula for entropy $S = \beta^2(\partial F/\partial\beta)$, we obtain following expression for quantum corrections to the Bekenstein–Hawking entropy for general static black holes:

$$S = \frac{2\pi^2}{45\beta^3}\int d\varphi\, d\theta \int_{r_+ + h}^{L} \frac{\sqrt{-g}}{g_{tt}^2}\, dr \tag{8}$$

We now consider some particular examples

2.1. The Schwarzschild Black Hole

Substituting the metric of the Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{9}$$

into equation (8) and then integrating over θ, φ, and r, we find the leading behavior of the entropy

$$S_{Sch} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{r_h}{360h} + \frac{1}{90}\ln\left(\frac{L}{h}\right) \tag{10}$$

where $r_h = 2M$ is the event horizon and $\beta = 4\pi r_h$ is the Hawking inverse temperature. The first part in equation (10) is the usual contribution from the vacuum surrounding the system at large distances. The second part is an intrinsic contribution from the horizon and it diverges linearly as $h \to 0$. The third part is a logarithmically divergent term. When we let $\delta^2 = 2\epsilon^2/15$ and $\Lambda^2 = L\epsilon^2/h = 30 r_h L$ [where $\delta = \int_{r_h}^{r_h+h}\sqrt{g_{rr}}\, dr \approx 2\sqrt{r_h h}$ is the proper distance from the horizon r_h to $r_h + h$, ϵ is the ultraviolet cutoff, and Λ is the infrared cutoff of Solodukhin(1995b), we can rewrite the entropy (10) as

$$S_{\text{Sch}} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{A_h}{48\pi\varepsilon^2} + \frac{1}{45}\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{11}$$

where $A_h = 4\pi r_h^2$ is area of the event horizon.

2.2. The Reissner–Nordström Black Hole

The metric of the Reissner–Nordström (RN) black hole is described by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - 2M/r + Q^2/r^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{12}$$

Inserting metric (12) into (8) and then taking the integration, we get

$$S_{RN} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{r_+ - r_-}{360h} + \left(-\frac{1}{180} + \frac{1}{60}\frac{r_+ - r_-}{r_+}\right)\ln\left(\frac{L}{h}\right) \tag{13}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and $\beta = 4\pi r_+^2/(r_+ - r_-)$. If we take $\delta^2 = 2\varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h = 30 r_+^2 L/(r_+ - r_-)$ [where $\delta = \int_{r_+}^{r_+ + h}\sqrt{g_{rr}}\, dr = 2\{[r_+^2/(r_+ - r_-)]h\}^{1/2}$ is the proper distance from the horizon r_+ to $r_+ + h$], then we can cast the entropy (13) as

$$S_{RN} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{A_h}{48\pi\varepsilon^2} + \left(-\frac{1}{90} + \frac{1}{30}\frac{4\pi r_+}{\beta}\right)\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{14}$$

where $A_h = 4\pi r_+^2$. For an extreme black hole [$M = Q$, $r_+ = M$, and $\beta(r_+) = \infty$], Equation (14) becomes

$$S_{\text{RN}} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{A_h}{48\pi\varepsilon^2} - \frac{1}{90}\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{15}$$

We know from equation (14) and (15) that the entropy of the RN black hole also contains three parts, like that of the Schwarzschild black hole.

2.3. The Garfinkle–Horowitz–Strominger Dilatonic Black Hole

By using the metric of the Garfinkle–Horowitz–Strominger (GHS) dilatonic black hole (Garfinkle *et al.*, 1991)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r(r - a)(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{16}$$

and equation (8), we obtain

$$S_{GHS} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{2M - a}{360h} + \left(\frac{1}{90} - \frac{a}{120r_+}\right) \ln\left(\frac{L}{h}\right) \tag{17}$$

where $r_+ = 2M$, $\beta = 4\pi r_+$, and $a = Q^2/2Me^{-2\phi_0}$. Noting that the proper distance from the horizon r_+ to $r_+ + h$ is $\delta = 2\sqrt{r_+ h}$, if we set $\delta^2 = \varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h = 30r_h L$, we can rewrite the entropy (17) as

$$S_{\mathrm{GHS}} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{A_h}{48\pi\varepsilon^2} + \left(\frac{1}{180} + \frac{1}{60}\frac{A_h}{r_+\beta}\right) \ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{18}$$

where $A_h = 4\pi r_+ (r_+ - a)$. In comparison with the results of Ghosh and Mitra (1994) we have an additional logarithmically divergent term

$$\left(\frac{1}{180} + \frac{1}{60}\frac{A_h}{r_+\beta}\right) \ln\left(\frac{\Lambda}{\varepsilon}\right)$$

which is dependent on the mass M and electric charge Q of the black bole and therefore cannot be neglected as a nonessential additive constant. Our results (17) and (18) are valid for both the extremal and nonextremal black holes. For the extremal black hole, i.e., $r_+ = 2M = a$ and $\beta(r_+) = 8\pi M$, the second part in (18) vanishes and the logarithmic term becomes

$$S^{\mathrm{ext}}_{\mathrm{GHS}} = \frac{1}{180} \ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{19}$$

which is whole entropy of the black hole if we ignore the contribution from the vacuum surrounding the system. In the extremal case our result is same as that of Ghosh and Mitra's (1994).

2.4. The Static Gibbons–Maeda Dilaton Black Hole

The metric of the static Gibbons–Maeda (GM) dilaton black hole is described by (Gibbons and Maeda, 1988)

$$ds^2 = -\frac{(r - r_+)(r - r_-)}{R^2} dt^2 + \frac{R^2 dr^2}{(r - r_+)(r - r_-)} + R^2(d\theta^2 + \sin^2\theta \, d\varphi^2), \tag{20}$$

where $r_+ = M \pm \sqrt{M^2 + D^2 - P^2 - Q^2}$, $D = (P^2 - Q^2)/2M$ and $R^2 = r^2 - D^2$. The parameters Q and P represent electric charge and magnetic charge, respectively.

The leading behavior of the entropy of a scalar field in the background of the black hole can be obtained by using (8) and (20), and is explicitly given by

$$S_{\mathrm{GM}} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{r_+ - r_-}{360h} + \left(-\frac{1}{180} + \frac{1}{60}\frac{r_+^2 - r_+r_-}{r_+^2 - D^2}\right)\ln\left(\frac{L}{h}\right) \quad (21)$$

here $\beta = 4\pi(r_+^2 - D^2)/(r_+ - r_-)$. When we set $\delta^2 = 2\varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h = 30(r_+^2 - D^2)L/(r_+ - r_-)$ (where $\delta = 2\{[(r_+^2 - D^2)/(r_+ - r_-)]h\}^{1/2}$), then we can rewrite the entropy (21) as

$$S_{\mathrm{GM}} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{A_h}{48\pi\varepsilon^2} + \left(-\frac{1}{90} + \frac{1}{30}\frac{4\pi r_+}{\beta}\right)\ln\left(\frac{\Lambda}{\varepsilon}\right) \quad (22)$$

where $A_h = 4\pi(r_+^2 - D^2)$.

In the case of a dilaton extreme black hole with only electric charge ($p = 0$), i.e., $M^2 = -Q^2/2$, and $\beta(r_+) = 8\pi M$, the horizon area vanishes, $A_h = 0$, if we take no notice of the contribution from the vacuum surrounding the system; then the whole black hole entropy is determined only by the logarithmically divergent term

$$S_{\mathrm{GM}}^{\mathrm{ext}} = \frac{1}{180}\ln\left(\frac{\Lambda}{\varepsilon}\right) \quad (23)$$

which is equal to the result of the extreme GM dilaton black hole.

2.5. The Garfinkle–Horne Dilaton Black Hole

The Garfinkle–Horne (GH) dilaton black hole metric in the Einstein–Maxwell dilaton theory can be expressed as (Garfinkle *et al.*, 1991; Horne and Horowitz, 1992)

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{(1-a^2)/(1+a^2)} dt^2 \\ &+ \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{(a^2-1)/(1+a^2)} dr^2 \\ &+ r^2\left(1 - \frac{r_-}{r}\right)^{2a^2/(1+a^2)} (d\theta^2 + \sin^2\theta\, d\varphi^2) \end{aligned} \quad (24)$$

with dilaton field $e^{2\Phi} = (1 - r_-/r)^{2a/(1+a^2)} e^{-2\Phi_0}$ and Maxwell field $F = (Q/r^2)\, dt \wedge dr$, where a is a coupling constant. $r = r_+$ is the location of the event horizon. For $a = 0$, $r = r_-$ is the location of the inner Canchy horizon;

however, for $a > 0$ the surface $r = r_-$ is singular. The mass M and charge Q of the black hole are related to parameters r_+ and r_- by

$$2M = r_+ + \left(\frac{1 - a^2}{1 + a^2}\right) r_- \qquad \text{and} \qquad Q^2 = \frac{r_+ r_-}{1 + a^2} e^{2a\Phi_0}$$

Substitution of (24) into (8) yields

$$S_{\rm GH} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{1}{360}\left[\frac{r_+ - r_-}{h} + \left(\frac{2(2a^2 - 1)}{1 + a^2} + \frac{6}{1 + a^2}\frac{A_k}{r_+\beta}\right)\ln\frac{L}{h}\right] \tag{25}$$

where

$$\beta = \frac{2\pi}{\kappa} = \frac{4\pi r_+}{(1 - r_-/r_+)^{(1-a^2)/(1+a^2)}}$$

$$A_h = \int \sqrt{g_{\theta\theta} g_{\varphi\varphi}}\, d\theta\, d\varphi = 4\pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)^{2a^2/(1+a^2)}$$

we let $\delta^2 = 2\varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h$, where

$$\delta = \int_{r_+}^{r_+ + h} \sqrt{g_{rr}}\, dr \approx 2r_+^{1/(1+a^2)} (r_+ - r_-)^{(a^2-1)/2(1+a^2)} \sqrt{h})$$

Then we can rewrite the entropy (25) as

$$S_{\rm GH} = \frac{8\pi^3 L^3}{135\beta^3} + \frac{A_h}{48\pi\varepsilon^2} + \frac{1}{90}\left(\frac{2a^2 - 1}{1 + a^2} + \frac{3}{1 + a^2}\frac{A_h}{r_+\beta}\right)\ln\frac{\Lambda}{\varepsilon} \tag{26}$$

In the extreme case $r_+ = r_-$, $(a \neq 0)$, the area of the event horizon vanishes. If we ignore the contribution from the vacuum surrounding the system, the whole black hole entropy is determined only by the logarithmically divergent term

$$S_{\rm GH}^{\rm ext} = \left[\frac{1}{45} - \frac{1}{30}\frac{1}{1 + a^2}\right]\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{27}$$

from which we know that $S_{\rm GH}^{\rm ext} < 0$ if $a^2 < 1/2$, $S_{\rm GH}^{\rm ext} = 0$ when $a^2 = 1/2$, and $S_{\rm GH}^{\rm ext} > 0$ for $a^2 > 1/2$.

3. THE ENTROPY OF QUANTUM SCALAR FIELD IN STATIC BLACK HOLES OBTAINED BY USING THE PATH INTEGRAL APPROACH

By comparing (11) and (14) here with (3) and (20) in Solodukhin (1995a) we find that the entropy of the Schwarzschild black hole and the RN black hole obtained by using the BWM are equal to the results obtained by EPIA if we ignore the contribution from the vacuum surrounding the system. To see if the two approaches give the same results for other static black holes, we calculate the quantum entropy of the black holes listed in Section 2 by using the EPIA.

We note that the formula (15) in Solodukhin (1995b) is only valid for the spacetime

$$ds^2 = -f(r)\, dt^2 + \frac{1}{f(r)}\, dr^2 + r^2\, \tilde{g}_{ij}(\theta) d\theta^i\, d\theta^j \tag{28}$$

The Schwarzschild black hole and the RN black hole possess this form, but the GM (Gibbons and Maeda, 1998), GHS, and GH black holes do not. The metrics of the black holes listed in Section 2 take the form

$$ds^2 = -f(r)\, dt^2 + \frac{1}{f(r)}\, dr^2 + R(r)^2 \tilde{g}_{ij}(\theta)\, d\theta^i\, d\theta^j \tag{29}$$

After we Euclideanize the metric (29) by setting $t \to ir$ and then let $d\tau = \beta\, d\phi$, we have

$$ds^2 = \beta^2 f(r)\, d\phi^2 + \frac{1}{f(r)}\, dr^2 + R(r)^2 \tilde{g}_{ij}(\theta)\, d\theta^i\, d\theta^j \tag{30}$$

We now follow Solodukhin (1995a, b) in assuming that the matter action of the scalar is $I_{\text{mat}} = \frac{1}{2} \int (\nabla\Phi)^2 \sqrt{-g} d^4x$, where g is the determinant of the metric and Φ is a massless scalar field. The contribution to the entropy of the black holes due to the matter fluctuations is given by

$$S^q = (\beta\partial_\beta - 1) I_{\text{eff}}(\beta, \Delta)|_{\beta=\beta_H} \tag{31}$$

where $\Delta = \nabla_\mu \nabla^\mu$ is the Laplace operator, and $I_{eff}(\beta, \Delta) = \frac{1}{2} \ln(det \Delta_{g\beta})$ is the one-loop effective action calculated in the classical spacetime background with conical singularity at the event horizon. The logarithm of the determinant is described by $\ln(det\, \Delta_{g\beta}) = -\int_{\varepsilon^2}^{\infty} s^{-1}\, Tr(e^{-s\Delta})$, where the integral over s is cut on the lower limit under $\varepsilon^2 = L^{-2}$, and L is the maximal impulse. Expanding $Tr(e^{-s\Delta})$ asymptotically in four dimensions and using the above results, we find the following divergent part of the effective action (Fursaev, 1994)

$$I_{\text{eff}}(\beta, \Delta) = -\frac{1}{32\pi^2}\left[\frac{1}{2} a_0 \varepsilon^{-4} + a_1 \varepsilon^{-2} + a_2 \ln\left(\frac{\Lambda}{\varepsilon}\right)^2\right] \tag{32}$$

where the coefficients take the form $a_n = a_n^{\text{reg}} + a_{\alpha,n}$. For the metric (30) the coefficients can be obtained by using the formulas of Fursaev (1994) and

Solodukhin (1995b). Inserting the coefficients into the effective action (32) and then using (31), we obtain the following quantum corrections to the Bekenstein–Hawking entropy:

$$S_q^I = \frac{A_h}{48\pi\varepsilon^2} + \left[\frac{a}{18} - \frac{A_h}{20\pi}\left(\frac{f(r)''|_{r_h}}{6} + \frac{\pi(R(r)^2)'|_{rh}}{R(r_h)^2\beta_h}\right)\right]\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{33}$$

with

$$a = \frac{1}{8\pi}\int_\Sigma R_\Sigma \sqrt{\gamma}\, d^2\theta$$

$$A_h = \int_\Sigma \sqrt{\gamma}\, d^2\theta$$

$$\beta_h = \frac{4\pi}{f(r)'|_{r_h}}$$

where we have used

$$\rho = \int \frac{dr}{\sqrt{f}}, \qquad \gamma_{ij} = R(r)^2\tilde{g}_{ij}, \qquad h_{ij} = \frac{1}{2\beta(r_h)}\left(\frac{dR^2}{dr}\right)_{r_h}\tilde{g}_{ij}$$

and primes denote differentiation with respect to r. The significance of the signs given here can be found in Solodukhin (1995b). Equation (33) is equal to (15) in Solodukhin (1995b) if $R^2(r_h) = r^2$ and $(1/8\pi)\int_\Sigma R_\Sigma\sqrt{\gamma}d^2\theta = 1$.

Since the results for the Schwarzschild black hole and RN black hole have been given in Solodukhin (1995a, b), in following we will use (33) to obtain the entropy for the GHS, the GM, and the GH dilaton black holes.

Inserting the metric (16) of the GHS dilatonic black hole (Garfinkle *et al.* 1991) into (33), we obtain

$$S_{\rm GHS}^I = \frac{A_h}{48\pi\varepsilon^2} + \left(\frac{1}{180} + \frac{1}{60}\frac{A_h}{r_+\beta}\right)\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{34}$$

Using the metric (20) of the static GM dilaton black hole (Gibbons and Maeda, 1988) and (33) we have

$$S_{\rm GM}^I = \frac{A_h}{48\pi\varepsilon^2} + \left(-\frac{1}{90} + \frac{1}{30}\frac{4\pi r_+}{\beta}\right)\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{35}$$

Substitution of the metric (24) of the GH dilaton black hole into (33) yields

$$S^I_{\rm GH} = \frac{A_h}{48\pi\varepsilon^2} + \frac{1}{90}\left(\frac{2a^2-1}{1+a^2} + \frac{3}{1+a^2}\frac{A_h}{r_+\beta}\right)\ln\frac{\Lambda}{\varepsilon} \tag{36}$$

If we ignore the contribution from the vacuum surrounding the system, the expressions (34)–(36) are equal to the results (18), (22), and (26), respectively.

4. CONCLUSION AND DISCUSSION

We investigated the entropy of the quantum scalar field for a minimally coupled quantum scalar field in thermal equilibrium at temperature $1/\beta$ in static black hole spacetime. The formula (8) was obtained by using the BWM and is valid for general static black holes. The equation (33) was found using the EPIA and is suitable for black holes which take the form of metric (29). Some examples were considered. It was shown that if we ignore the contribution from the vacuum surrounding the system and take $\delta^2 = 2\varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h$, then the entropy obtained by using the BWM is equal to that obtained by the EPIA for the black holes listed in this paper. The entropy of a static black hole consists of two parts: a quadratically divergent term and a logarithmically divergent term.

The quadratic parts take the geometric character $A_h/48\pi\varepsilon^2$, which can be regarded as a renormalization of the gravitational constant as discussion in Solodukhin (1995b, c) and Ghosh and Mitra (1994). The renormalized gravitational constant is determined by $1/G_{\rm ren} = 1/G + 1/(12\pi\varepsilon^2)$.

The logarithmically divergent terms are not proportional to the horizon area. For the Schwarzschild black hole the term is not essential and does not influence the physics since we know that the entropy is generally defined up to an arbitrary additive constant. But for the RN black hole, the GHS dilatonic black hole, the GM dilaton black hole, and the GH dilaton black hole the terms depend on the characteristics of the black holes (mass M, charge Q, etc.), and therefore cannot be neglected as nonessential additive constants. For nonextremal black holes, if we define the quantum-corrected radius of horizon $r^2_{h,\rm ren} = r^2_h + \xi l^2_{\rm PL}$, (where $l^2_{\rm PL} = G_{\rm ren}$ is the Planck length), the quantity $\xi = \xi(M, Q, \text{etc.}) \ln(\Lambda/\varepsilon)$ absorbs the logarithmic divergence of the entropy. The parameters $\xi(m, Q, \text{etc.})$ of the black holes are (a) for the Schwarzschild black hole, $\xi(M) = 1/45\pi$, (b) for the RN black hole $\xi(M, Q) = (-1/90\pi + (1/30\pi)4\pi r_+/\beta)$; (c) for the GHS dilatonic black hole $\xi(M, Q) = (1/180\pi + (1/60\pi)A_b/r_+\beta)$; (d) for the GM dilaton black hole $\xi(M, Q) = (-1/90\pi + (1/30\pi)4\pi r_+/\beta)$; and (e) for the GH dilaton black hole

$$\xi(M, Q) = \frac{1}{90\pi}\left(\frac{2a^2 - 1}{1 + a^2} + \frac{3}{1 + a^2}\frac{A_\Sigma}{r_+\beta_H}\right)$$

The quantum corrections will increase the horizon radius if $\xi(M, Q) > 0$, but will decrease the radius if $\xi(M, Q) < 0$.

In the case of the extremal black hole, the logarithmically divergent terms of the RN black hole ($r_+ = M = Q$) can be renormalized as in nonextremal case. But for the extreme GHS dilatonic black hole ($r_h = 2M = a$), the extreme GM dilaton black hole with only electric charge ($M^2 = Q^2/2$), and the extreme GH dilaton black hole ($r_+ = r_-$), the area of the horizon vanishes, and the whole black hole entropy is determined by the logarithmically divergent term. The logarithmically divergent terms cannot be renormalized by the gravitational constant because even with the renormalization, the zero area should make the entropy vanish.

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